JOURNAL OF NUMBER THEORY **59**, 319–351 (1996)
ARTICLE NO. 0101

On the q-Analogue of the Pair Correlation Conjecture

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Communicated by Alan C. Woods

Received February 15, 1991; revised May 24, 1993

Let

where the inner-most sum runs over the imaginary parts γ of zeros of Dirichlet L-functions $L(s, \chi)$ and where

$$N_K(Q) = \frac{Q \log Q}{2\pi} \int_{-\infty}^{+\infty} \left(K\left(\frac{1}{2} + it\right)\right)^2 dt,$$

and $K(s)$ is a suitably chosen "kernel." Under the assumption of the Generalized Riemann Hypothesis we estimate the asymptotic size of $F_K(\alpha, Q)$ as $Q \to +\infty$ in the range $|\alpha| < 2$. This is used to prove that the proportion of simple zeros of all Dirichlet L-functions is greater than or equal to $\frac{11}{12}$ in the sense of the inequality

$$\frac{1}{N_K(Q)} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma \\ \text{simple}}} \left(K\left(\frac{1}{2} + i\gamma\right)\right)^2 \geqslant \frac{11}{12} + o(1)$$

as $Q \to +\infty$.

1. INTRODUCTION

The study of the pair correlation of the zeros of the Riemann zeta function $\zeta(s)$ was initiated by Montgomery in [14]. Since then, numerous papers (see, for example [3–5, 16, 18] and references therein.) have appeared on this topic, resulting in various applications to the distribution of primes and to the vertical distribution of the zeros of various zeta functions.

0022-314X/96 $18.00

Montgomery's original idea was to investigate the function

$$F(\alpha, T) = \left(\frac{T \log T}{2\pi}\right)^{-1} \sum_{\substack{0 < \gamma \leqslant T \\ 0 < \gamma' \leqslant T}} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \tag{1}$$

where $\alpha, T \in \mathbb{R}$, $T \geqslant 2$, $w(u) = 4/(4+u^2)$ and $\zeta(\frac{1}{2} + i\gamma) = \zeta(\frac{1}{2} + i\gamma') = 0$. Equivalently, it can easily be seen that

$$F(\alpha, T) = \frac{4}{T \log T} \int_{-\infty}^{+\infty} \left| \sum_{0 < \gamma \leqslant T} k(t, \gamma)\, T^{i\alpha\gamma} \right|^2 dt, \tag{2}$$

where

$$k(t, \gamma) = \frac{1}{1 + (t - \gamma)^2}. \tag{3}$$

Under the Riemann Hypothesis, Montgomery's analysis has led to the estimate

$$F(\alpha, T) = T^{-2|\alpha|} \log T + |\alpha| + O\left(|\alpha|\, T^{|\alpha| - 1} + T^{-3/2\,|\alpha|} + \frac{1}{\log T}\right) \tag{4}$$

as $T \to +\infty$, uniformly on each interval $|\alpha| \leqslant 1 - \varepsilon$, and to the conjecture,

$$F(\alpha, T) = 1 + o(1) \text{ as } T \to +\infty \text{ in the range } 1 \leqslant a \leqslant |\alpha| \leqslant b < +\infty, \tag{5}$$

for any constants a, b.

The assertion in (5) implies the pair correlation conjecture:

$$\left(\frac{T \log T}{2\pi}\right)^{-1} \left| \left\{ (\gamma, \gamma') \colon 0 < \gamma, \gamma' \leqslant T, \frac{2\pi\alpha}{\log T} \leqslant \gamma - \gamma' \leqslant \frac{2\pi\beta}{\log T} \right\} \right|$$

$$\sim \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du, \qquad \text{as} \quad T \to +\infty. \tag{6}$$

In this paper, we study the q-analogue of $F(\alpha, T)$ defined in (2). For this purpose, we define

$$F_K(\alpha, Q) = \frac{1}{N_K(Q)} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \left| \sum_{\gamma} K\left(\frac{1}{2} + i\gamma\right) Q^{i\alpha\gamma} \right|^2, \tag{7}$$

where the inner-most sum runs over the imaginary parts γ of zeros of the Dirichlet L-function $L(s, \chi)$, and where

$$N_K(Q) = \frac{Q \log Q}{2\pi} \int_{-\infty}^{+\infty} \left(K\left(\frac{1}{2} + it\right)\right)^2 dt. \tag{8}$$

Here the "kernel" $K(s)$ is an analytic function defined in a strip D containing the critical strip, $K(\sigma+it)\log(|t|+2)\in L^1(-\infty,+\infty)$ for all fixed $\sigma\in D\cap\mathbb{R}$, and $K(\frac{1}{2}+it)=K(\frac{1}{2}-it)$ is real for all real t. We assume the Generalized Riemann Hypothesis (G.R.H.) and the following restriction on the choice of $K(s)$: The Mellin transform $a(u)$ of $K(s)$,

$$a(u)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)\,u^{-s}\,ds \tag{9}$$

exists for $c\in D\cap\mathbb{R}$, has compact support $\operatorname{Supp} a(u)\subset[A,B]\subset\mathbb{R}^+$ and is of bounded total variation. We note that

$$K(s)=\int_0^\infty a(u)\,u^{s-1}\,du.$$

A particular choice of $K(s)$, $K(s)=(e^{s-1/2}-e^{-s+1/2}/2s-1)^2$ has been used in [17]. We remark that in this particular case $K(\frac{1}{2}+i\gamma)=(\sin\gamma/\gamma)^2$ and $N_K(Q)=\frac{1}{3}Q\log Q$. We also note that the "normalization factor" $N_K(Q)$ is actually the asymptotic size of the diagonal terms in (7).

$$\sum_{q\leqslant Q}\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\sum_{\gamma}\left(K\left(\frac{1}{2}+i\gamma\right)\right)^2\sim N_K(Q)\qquad\text{as}\quad Q\to\infty, \tag{10}$$

which can be obtained by partial summation and by known results on the density of zeros on the critical line. We remark that $F_K(\alpha,Q)$ is an even function of α and is non-negative. We now state our main theorem on $F_K(\alpha,Q)$ which was announced in [18].

MAIN THEOREM. *If G.R.H. holds then*

$$F_K(\alpha,Q)=\begin{cases}\delta_Q(\alpha)\left(1+O\left(\dfrac{1}{\log Q}\right)\right)+|\alpha|+O\left(\dfrac{1}{\log Q}\right)\\ \qquad\text{if}\quad |\alpha|\leqslant 1\\ 1+O\left(\dfrac{1}{\log Q}\right)\\ \qquad\text{if}\quad 1\leqslant|\alpha|\leqslant 2-14\,\dfrac{\log\log Q}{\log Q}\end{cases} \tag{11}$$

uniformly as $Q\to+\infty$, where

$$\delta_Q(\alpha)=\frac{Q^{1-\alpha}a^2(Q^\alpha)\log^2 Q}{N_K(Q)}. \tag{12}$$

We also have

COROLLARY. *If G.R.H. holds, the proportion of simple zeros of all Dirichlet L-functions is greater than or equal to* $\frac{11}{12}$ *in the sense of the inequality*

$$\frac{1}{N_K(Q)} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma \\ \text{simple}}} \left(K\left(\frac{1}{2}+i\gamma\right)\right)^2$$

$$\geqslant \frac{11}{12} + o(1) \qquad \text{as} \quad Q \to +\infty.$$

Clearly, (11) is in support of the conjecture (5) for Dirichlet L-functions. The remainder of this paper is denoted to proving assertions (11) and (13). The function $\delta_Q(\alpha)$ in (12) behaves like a Dirac δ-function. By (8), (9), and (12) we see that as $Q \to +\infty$,

$$\delta_Q(0) \sim \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(K\left(\frac{1}{2}+it\right)\right)^2 dt\right)^{-1} a^2(1) \log Q, \tag{14}$$

and for fixed $\alpha \neq 0$, $\delta_Q(\alpha) = 0$ by compact support of $a(u)$. Furthermore,

$$\int_{-\infty}^{+\infty} \delta_Q(\alpha)\, d\alpha = 1 \qquad \text{for all } Q. \tag{15}$$

We note that Plancherel's Theorem for the Mellin transform in the form

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(K\left(\frac{1}{2}+it\right)\right)^2 dt = \int_0^{+\infty} a^2(u)\, du \tag{16}$$

is used to obtain (15).

2. NOTATION AND PRELIMINARIES

In applications of the Hardy–Littlewood–Vinogradov circle method in Section 3, we use the following dissection of the unit interval and the associated notation:

If $b'/r' < b/r < b''/r''$ are consecutive Farey fractions of order R, we let $\mathscr{M}(r, b)$ denote the interval $[b+b'/r+r', b+b''/r+r'')$, the Farey arc around b/r. We note that

$$\frac{b}{r} - \frac{b+b'}{r+r'} = \frac{1}{r(r+r')} \tag{17}$$

and that

$$R < r + r' \leqslant 2R, \tag{18}$$

so that we have

$$\left[\frac{b}{r} - \frac{1}{2rR}, \frac{b}{r} + \frac{1}{2rR}\right] \subseteq \mathscr{M}(r, b) \subseteq \left[\frac{b}{r} - \frac{1}{rR}, \frac{b}{r} + \frac{1}{rR}\right]. \tag{19}$$

In the established terminology of the circle method, $\mathscr{M}(r, b)$'s correspond to "major arcs", and we have no "minor arcs". We will denote by $M(r, b)$ the Farey arcs shifted to the origin:

$$M(r, b) = \left[\frac{1}{r(r+r')}, \frac{1}{r(r+r'')}\right] \subset \left[\frac{-1}{rR}, \frac{1}{rR}\right]$$

We also use the following notation throughout this paper:

p	prime number
m, n, k, q	positive integers
$\chi(n)$	character modulo q
χ_o	principal character
$[x]$	greatest integer not exceeding x
$\Vert x \Vert$	distance from x to the nearest integer
$\mu(n)$	Mobius function
$\phi(n)$	Euler's function
$\Lambda(n)$	von Mangoldt function
$\sum_{\chi}$	sum over all characters modulo q
$L(s, \chi)$	Dirichlet L-function
$\frac{1}{2} + i\gamma$	A typical zero of $L(s, \chi)$

(γ represents Euler's constant in one occasion, but there the context makes it clear.)

$$\sum_{a=1}^{q}{}^{*} = \sum_{\substack{a=1 \\ (a,q)=1}}^{q}$$

$$e(\alpha) = e^{2\pi i \alpha}$$

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m)\, e\left(\frac{m}{q}\right)$$

$$\kappa = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(K\left(\frac{1}{2} + it\right)\right)^2 dt.$$

The familiar notations $O(\ )$, $o(\ )$, $\sim$, $\ll$ and $\ll_\theta$ are also used; here the subscript θ indicates the dependence of the implicit constant on θ. We also use the notation

$$J(x; Q, z) = \sum_{Q < q \leqslant z} \sum_{\substack{m < n \\ m \equiv n \,(\mathrm{mod}\, q)}} a\left(\frac{m}{x}\right) a\left(\frac{n}{x}\right) \Lambda(m)\, \Lambda(n) \tag{20}$$

$$S(x, \alpha) = \sum_m a\left(\frac{m}{x}\right) \Lambda(m)\, e(m\alpha), \tag{21}$$

where $a(u)$ is defined in (9) and

$$W(x, \alpha) = \sum_{\substack{1 \leqslant k \leqslant Bx/q \\ Q < q \leqslant x}} e(-qk\alpha), \tag{22}$$

where B is from $\mathrm{Supp}\, a(u) \subset [A, B]$.

We remark that

$$J(x; Q, x) = \int_0^1 |S(x, \alpha)|^2\, W(x, \alpha)\, d\alpha. \tag{23}$$

We now record several lemmas that are used in this paper. The proofs can be found respectively in [18, 20, 2] for Lemmas 1 through 3. Otherwise they are given in this paper.

LEMMA 1. *Assume GRH. For* $x \geqslant 1$, *and any character* $\chi\, (\mathrm{mod}\, q)$, *we have*

$$\sum_\gamma K\left(\frac{1}{2} + i\gamma\right) x^{i\gamma} = E(\chi)\, K(1)\, x^{1/2} - x^{-1/2} \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n)\, \chi(n)$$
$$+ x^{-1/2} a\left(\frac{1}{x}\right) \log \frac{q}{\pi} + O(\min(x^{1/2}, x^{-1/2} \log q \log x)), \tag{24}$$

where $E(\chi) = 0$ *or* 1 *according as* $\chi \neq \chi_o$ *or* $\chi = \chi_o$, *and where* γ *ranges over non-trivial zeros of* $L(s, \chi)$. *The implicit constant in* (24) *depends only on the kernel* K, *and should be interpreted as* $O(1)$ *when* $x = 1$.

LEMMA 2. *Let* $r(t)$, $\theta'(t) = d\theta/dt$ *be continuous real valued functions. If* $|r(t)/\theta'(t)| \leqslant \lambda$ *and the variation* $\mathrm{Var}(r(t)/\theta'(t)) \leqslant 2\lambda$, *then*

$$\left| \int_a^b r(t)\, e^{i\theta(t)}\, dt \right| \leqslant 4\lambda. \tag{25}$$

LEMMA 3 (Gallagher). *Let $S(t)=\sum_\nu c(\nu)\,e(\nu t)$ be an absolutely convergent exponential sum; with $\nu\in\mathbb{R}$ and $c(\nu)\in\mathbb{C}$. For $0<\theta<1$, put $\delta=\theta/T$. Then*

$$\int_{-T}^{T} |S(t)|^2\,dt \ll_\theta \int_{-\infty}^{+\infty} \left|\delta^{-1}\sum_{x}^{x+\delta} c(\nu)\right|^2 dx \tag{26}$$

LEMMA 4. *Let*

$$\Psi^*(x,\chi)=\begin{cases}\sum_{n\leqslant x}\Lambda(n)\,\chi(n) & \text{if } \chi\neq\chi_0\\ \sum_{n\leqslant x}\Lambda(n)\,\chi_0(n)-x & \text{if } \chi=\chi_0\end{cases} \tag{27}$$

We have for $x\geqslant 2$, $T\geqslant 2$,

$$\Psi^*(x,\chi)=-\sum_{|\gamma|\leqslant T}\frac{x^\rho}{\rho}+R_1(x,\chi)+R_2(x,\chi),$$

where

$$\int_2^x |dR_1| \ll \log qx, \tag{28}$$

$$R_2 \ll \log x+\frac{x}{T}(\log x)^2, \tag{29}$$

and

$$\int_2^x |R_2(x)|\,dx \ll \frac{x^2}{T}(\log x)^2. \tag{30}$$

Proof. Let m be a non-negative integer, $c=1+(\log x)^{-1}$, $x\geqslant 2$, $U=a+m+1$, and $a=0$ or 1 according to $\chi(-1)=(-1)^a$. Assume first that χ is primitive (mod q). We consider the contour C composed of

C_1: the line segment $[c-iT,\ -U-iT]$,

C_2: the line segment $[-U+iT, c+iT]$, and

C_3: the line segment $[-U-iT,\ -U+iT]$.

Here it is possible to choose T in a way [1, p. 120] to ensure that $(L'/L)(s,\chi)\ll \log^2 qT$ on C. Let

$$J(x,\chi,T)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{L'}{L}(x,\chi)\frac{x^s}{s}\,ds. \tag{31}$$

By the lemma of Chapter 17 of Davenport [1], we have

$$\Psi^*(x, \chi) = J(x, \chi, T) + R_0(x, T), \tag{32}$$

where

$$R_0(x, T) = \sum_{\substack{n=1 \\ n \neq x}}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c \min\left(1, T^{-1} \left|\log \frac{x}{n}\right|^{-1}\right) + cT^{-1}\Lambda(x). \tag{33}$$

We now observe that the only poles of the integrand are at the zeros of $L(s, \chi)$ and at $s = 0$. About $s = 0$, the following expansion holds true:

$$\frac{L'}{L}(s, \chi) = \frac{1-a}{s} + b_0(\chi) + b_1(\chi)\, s + \cdots, \tag{34}$$

where b_0 and b_1 are constants depending on χ. We note that the trivial zeros of $L(s, \chi)$ are at $s = -(2n + a)$, $n = 0, 1, 2, \ldots$, and we let, as usual, ρ represent a non-trivial zero of $L(s, \chi)$. On applying Cauchy's residue theorem to the integral in (31) and making use of (32) we obtain

$$\begin{aligned} \Psi^*(x, \chi) = &- \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} - (1-a) \log x - b_0(\chi) \\ &+ \sum_{0 < 2n - a < U} \frac{x^{a-2n}}{2n-a} + \frac{1}{2\pi i} \int_C -\frac{L'}{L}(s, \chi) \frac{x^s}{s}\, ds + R_0(x, T). \end{aligned} \tag{35}$$

To complete the proof of our lemma, we let $U \to \infty$, and observe that the integral along C_3 tends to zero. We set

$$R_1(x, \chi) = (1-a) \log x + b_0(\chi) + \sum_{2m - a > 0} \frac{x^{a-2m}}{2m-a}, \tag{36}$$

and

$$R_2(x, \chi) = R_0(x, T) + \frac{1}{2\pi i} \int_{C_1 \cup C_2} -\frac{L'}{L}(s, \chi) \frac{x^s}{s}\, ds. \tag{37}$$

We note that $R_1(x, \chi)$ is monotonically increasing and that its variation can easily be calculated to confirm the validity of (28). The estimate (29) is derived in Davenport [1]. The derivation there also shows that the appearance of the $\log x$ term on the right side of (29) is due to the largest prime power x less than x, $\frac{3}{4}x < x_1 < x$, or the least prime power x_2 larger than x, $x < x_2 < \frac{5}{4}x$, the actual error term being $\log x \min(1, x/(T\langle x \rangle))$, where $\langle x \rangle$ represents the distance from x to the nearest prime power other than x itself. In obtaining (30) therefore, we only estimate the contribution of these "peaks" around prime powers, the other terms contributing trivially an amount that is $\ll x \cdot x/T(\log x)^2$. It is easily seen that the width of each peak is $\ll x/T$ and therefore the total contribution due to prime powers is

$$\ll \frac{x}{T} \log x \sum_{n < 5/4x} \Lambda(n)$$

$$\ll \frac{x^2}{T} (\log x)^2.$$

Finally, we remark that in the case of an imprimitive character, the error caused can be absorbed into R_1. This completes the proof of the lemma.

Lemma 5. *We have*

$$\text{(a)} \quad \sum_{k \leqslant x} \frac{1}{\phi(k)} = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} \log x + c_1 + O\left(\frac{\log x}{x}\right). \tag{38}$$

$$\text{(b)} \quad \sum_{k \leqslant x} (x-k) \frac{1}{\phi(k)} = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} x \log x + c_2 x + \frac{1}{2} \log x + c_3 + E(x), \tag{39}$$

where c_1, c_2 *and* c_3 *are constants and*

$$c_1 = c_2 + \frac{\zeta(2)\,\zeta(3)}{\zeta(6)}, \tag{40}$$

and

$$E(x) \ll x^{-1/4}. \tag{41}$$

The result (a) is due to Landau [15]. A stronger result with the error reduced to $O((\log x)^{2/3}/x)$ has been obtained by Sitaramachandrarao [19]. The result (b) may be proved by the same method as used by Hooley [7] to prove a similar result.

We now prove an identity that will find application in Section 4.

Lemma 6. *We have*

$$\frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ (\bmod q)}} \left| \sum_n a\left(\frac{n}{x}\right) \Lambda(n)\, \chi(n) \right|^2 = S_1(x, q) + S_2(x, q) - S_3(x, q), \tag{42}$$

where

$$S_1(x, q) = 2 \sum_{\substack{m<n \\ m \equiv n (\mathrm{mod}\, q) \\ (mn, q)=1}} a\left(\frac{m}{x}\right) a\left(\frac{n}{x}\right) \Lambda(m)\, \Lambda(n), \tag{43}$$

$$S_2(x, q) = \sum_{(n, q)=1} a^2\left(\frac{n}{x}\right) \Lambda^2(n), \tag{44}$$

$$S_3(x, q) = \frac{1}{\phi(q)} \left(\sum_{(n, q)=1} a\left(\frac{n}{x}\right) \Lambda(n) \right)^2. \tag{45}$$

Proof. Using the orthogonality of characters, we see that

$$\frac{1}{\phi(q)} \sum_{\chi} \left| \sum_n a\left(\frac{x}{n}\right) \Lambda(n)\, \chi(n) \right|^2 = \sum_{\substack{h=1 \\ (h, q)=1}}^{q} \left| \sum_{n \equiv h (\mathrm{mod}\, q)} a\left(\frac{n}{x}\right) \Lambda(n) \right|^2, \tag{46}$$

and on expanding out the square, we find that this is

$$\sum_{\substack{m \equiv n\ (\mathrm{mod}\, q) \\ (mn, q)=1}} a\left(\frac{m}{x}\right) a\left(\frac{n}{x}\right) \Lambda(m)\, \Lambda(n).$$

Here $S_1(x, q)$ and $S_2(x, q)$ arise respectively from terms with $m \neq n$ and $m = n$; and $S_3(x, q)$ represents the terms belonging to $\chi = \chi_0$ and that are excluded from the sum on the left of (42). This completes the proof.

In Section 4, we estimate an expression of the form

$$J(x; y, z) = \sum_{y<q \leqslant z} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0\ (\mathrm{mod}\, q)} \left| \sum_n a\left(\frac{n}{x}\right) \Lambda(n)\, \chi(n) \right|^2 \tag{47}$$

for certain specified ranges of x, y and z. By means of the above lemma, we have

$$J(x; y, z) = J_1(x; y, z) + J_2(x; y, z) - J_3(x; y, z) \tag{48}$$

where

$$J_i(x; y, z) = \sum_{y<q \leqslant z} S_i(x, q); \qquad i = 1, 2, 3 \tag{49}$$

and we estimate each term on the right of (48) separately.

LEMMA 7. *Assume G.R.H. Let*

$$\Psi(x, \chi, \beta) = \begin{cases} \sum_{m \leqslant x} \Lambda(m)\, \chi(m)\, e(m\beta) & \text{if } \chi \neq \chi_0, \\ \sum_{m \leqslant x} \Lambda(m)\, \chi(m)\, e(m\beta) - \sum_{m \leqslant x} e(m\beta) & \text{if } \chi = \chi_0 \end{cases}. \tag{50}$$

Then for any $\delta > 0$ *and any character* $\chi(\text{mod } q)$.

$$\int_{-\delta}^{+\delta} |\Psi(x, \chi, \beta)|^2 \, d\beta \ll \delta x (\log qx)^4. \tag{51}$$

Proof. We have by Lemma 4,

$$\Psi(x, \chi, \beta) = \int_{2^-}^{x} e(t\beta) \, d(\Psi^*(t, \chi))$$

$$= - \sum_{|\gamma| \leqslant T} \int_{2^-}^{x} e(t\beta) \, t^{\rho - 1} \, dt + \int_{2^-}^{x} e(t\beta) \, d(R_1 + R_2). \tag{52}$$

Set $T = x^2$. Appealing to Lemma 4 again, we obtain

$$\Psi(x, \chi, \beta) = - \sum_{|\gamma| \leqslant x^2} I(\rho, \beta) + O(\log^2 qx). \tag{53}$$

where

$$I(\rho, \beta) = \int_{2^-}^{x} e(t\beta) \, t^{\rho - 1} \, dt = \int_{2^-}^{x} t^{-1/2} e^{i(\gamma \log t + 2\pi t\beta)} \, dt.$$

We note that by integrating by parts in (52), we obtain

$$\Psi(x, \chi, \beta) \ll \max_{t \leqslant x} |\Psi^*(t, \chi)| \cdot (1 + |\beta| \, x)$$

$$\ll x^{1/2} \log^2 qx \tag{54}$$

for $|\beta| \leqslant x^{-1}$. Since this proves the lemma for the case $\delta \leqslant x^{-1}$, we can assume, without loss of generality, that $\delta > x^{-1}$. By (53) we have

$$\int_{-\delta}^{+\delta} |\Psi(x, \chi, \beta)|^2 \, d\beta \ll \delta (\log qx)^4 + \int_{-\delta}^{+\delta} \Bigg| \sum_{|\gamma| < x^2} I(\rho, \beta) \Bigg|^2 \, d\beta. \tag{55}$$

We now estimate the contribution of zeros in the range $7\delta x < |\gamma| \leqslant x^2$ to the integral on the right of the above. In this range the hypotheses of Lemma 2 are satisfied for $r(t) = t^{-1/2}$ and $\theta(t) = \gamma \log t + 2\pi t\beta$, and therefore we have

$$\int_{Y/2}^{Y} e(\beta t) \, t^{\rho - 1} \, dt \ll \frac{Y^{1/2}}{|\gamma|}, \tag{56}$$

implying

$$\int_{2^-}^{x} e(\beta t)\, t^{\rho-1}\, dt \ll \frac{x^{1/2}}{|\gamma|} \tag{57}$$

for $|\gamma| > 7\delta x$.

Hence, the right hand side of (55) is affected by an amount that is

$$\ll x \int_{-\delta}^{+\delta} \left| \sum_{7\delta x < |\gamma| \leqslant x^2} \frac{1}{|\gamma|} \right|^2 d\beta$$

$$\ll \delta x \log^4 qx.$$

It suffices, therefore to consider the integral

$$\int_{-\delta}^{+\delta} \left| \int_2^x e(\beta t) \left(\sum_{|\gamma| \leqslant 7\delta x} t^{\rho-1} \right) dt \right|^2 d\beta. \tag{58}$$

We extend the range of β to the entire real line, and deduce by Plancherel's identity that the above is

$$\leqslant \int_2^x \left| \sum_{|\gamma| \leqslant 7\delta x} t^{\rho-1} \right|^2 dt = \int_2^x \left| \sum_{|\gamma| \leqslant 7\delta x} t^{i\gamma} \right|^2 \frac{dt}{t}.$$

Putting $t = e^y$ and applying Gallagher's lemma (Lemma 3), we obtain

$$\int_{-\delta}^{+\delta} \left| \sum_{|\gamma| \leqslant 7\delta x} I(\rho, \beta) \right|^2 d\beta \ll (\log x)^2 \int_{-\infty}^{+\infty} \left| \sum_{\substack{|\gamma - t| < (\log x)^{-1} \\ |\gamma| \leqslant 7\delta x}} \right|^2 dt. \tag{59}$$

Here the sum is $\ll \log qx$, and vanishes for $|t| \gg \delta x$, so the above is $\ll \delta x\, (\log qx)^4$, as required.

LEMMA 8. *Assume G.R.H. Let*

$$\Psi^*(x, \chi, \beta) = \begin{cases} \sum_m a\left(\frac{m}{x}\right) \Lambda(m)\, \chi(m)\, e(m\beta) \\ \qquad \text{if} \quad \chi \neq \chi_0, \\ \sum_m a\left(\frac{m}{x}\right) \Lambda(m) \chi(m)\, e(m\beta) - \sum_m a\left(\frac{m}{x}\right) e(m\beta) \\ \qquad \text{if} \quad \chi = \chi_0. \end{cases} \tag{60}$$

Then for any $\delta > 0$ *and any character* $\chi \pmod q$,

$$\int_{-\delta}^{+\delta} |\Psi^*(x, \chi, \beta)|^2 \, d\beta \ll \delta x (\log qx)^4. \tag{61}$$

Proof. We have

$$\Psi^*(x, \chi, \beta) = \int_{Ax}^{Bx} a(t/x) \, d\Psi(t, \chi, \beta) = -\int_{Ax}^{Bx} \Psi(t, \chi, \beta) \, da(t/x).$$

Hence

$$\int_{-\delta}^{+\delta} |\Psi^*(x, \chi, \beta)|^2 \, d\beta = \int_{-\delta}^{+\delta} \left| \int_{Ax}^{Bx} \Psi(t, \chi, \beta) \, da\left(\frac{t}{x}\right) \right|^2 d\beta \tag{62}$$

By the Cauchy–Schwarz inequality the above is

$$\leqslant \int_{-\delta}^{+\delta} \left(\int_{Ax}^{Bx} |\Psi(t, \chi, \beta)|^2 \left| da\left(\frac{t}{x}\right) \right| \int_{Ax}^{Bx} \left| da\left(\frac{t}{x}\right) \right| \right) d\beta.$$

Since $a(t/x)$ is of bounded variation, this is

$$\ll \int_{-\delta}^{+\delta} \left(\int_{Ax}^{Bx} |\Psi(t, \chi, \beta)|^2 \left| da\left(\frac{t}{x}\right) \right| \right) d\beta.$$

We interchange the order of integration and see by Lemma 7 that this is

$$\ll \delta \int_{Ax}^{Bx} t (\log qt)^4 \left| da\left(\frac{t}{x}\right) \right|$$

$$\ll \delta x (\log qx)^4 \int_{Ax}^{Bx} \left| da\left(\frac{t}{x}\right) \right|$$

$\ll \delta x (\log qx)^4$, as required.

LEMMA 9. *Assume G.R.H. Let*

$$T(x, \beta) = \frac{1}{\phi(r)} \sum_{\chi \pmod r} \tau(\bar{\chi}) \, \chi(b) \, \Psi^*(x, \chi, \beta), \tag{63}$$

where $\Psi^*(x, \chi, \beta)$ *is defined as in* (60). *Then*

$$\sum_b{}^* \int_{-\delta}^{+\delta} |T(x, \beta)|^2 \, d\beta \ll r \delta x (\log rx)^4. \tag{64}$$

Proof. We have

$$\sum_b{}^* |T(x, \beta)|^2 = \frac{1}{\phi^2(r)} \sum_b{}^* \left| \sum_{\chi \pmod r} \tau(\bar{\chi}) \, \chi(b) \, \Psi^*(x, \psi, \beta) \right|^2. \tag{65}$$

By the orthogonality of characters, this is

$$=\frac{1}{\phi(r)}\sum_{\chi\,(\mathrm{mod}\,r)}|\tau(\bar{\chi})|^2\,|\Psi^*(x,\chi,\beta)|^2,$$

which is

$$\ll\frac{r}{\phi(r)}\sum_{\chi\,(\mathrm{mod}\,r)}|\Psi^*(x,\chi,\beta)|^2.$$

Hence

$$\sum_b{}^*\int_{-\delta}^{+\delta}|T(x,\beta)|^2\,d\beta\ll\frac{r}{\phi(r)}\sum_{\chi\,(\mathrm{mod}\,r)}\int_{-\delta}^{+\delta}|\Psi^*(x,\chi,\beta)|^2\,d\beta.$$

In order to complete the proof of the lemma, it now suffices to appeal to Lemma 8.

3. The Circle Method

In order to apply the circle method to our problem, we write (24) as $L(x,\chi)=R(x,\chi)$ and observe that by (7),

$$F_K(\alpha,Q)=\frac{1}{N_K(Q)}\sum_{q\leqslant Q}\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}|L(Q^\alpha,\chi)|^2.$$

This shows that an estimate of $F_K(\alpha,Q)$ would follow from that of $\sum_{q\leqslant Q}1/(\phi(q))\sum_{\chi\,(\mathrm{mod}\,q)}|R(Q^\alpha,\chi)|^2$. On expanding this latter expression, one finds that the crux of proving (11) lies in estimating $J(x;Q,x)$.

Our method of estimating $J(x;Q,x)$ depends, through equation (23), on applying the circle method to the integral

$$\int_0^1|S(x,\alpha)|^2\,W(x,\alpha)\,d\alpha. \tag{66}$$

To this end we now present a lemma the proof of which is based on an argument of Vaughan [21].

Lemma 10. *Let* $R=[x^{1/2}]$ *and*

$$W(x,\alpha)=\sum_{\substack{1\leqslant k\leqslant B(x/q)\\ Q<q\leqslant x}}e(-qk\alpha). \tag{67}$$

For $\alpha \in \mathcal{M}(r, b)$, *and* $R \ll Q$ *we have*

$$W(x, \alpha) \ll \frac{x}{r} \log x.$$

Proof. Let $\alpha = b/r + \beta \in \mathcal{M}(r, b)$. We have $|\beta| \leqslant 1/rR$ by (19). We first sum over q in (67) to see that

$$W(x, q) \ll \sum_{1 \leqslant k \leqslant Bx/Q} \min\left(\frac{x}{k}, \frac{1}{\|k\alpha\|}\right).$$

We now write $k = hr + s$ for $0 \leqslant h < Bx/Qr$ and $1 \leqslant s \leqslant r$. Then the above sum is

$$\ll \sum_{0 \leqslant h < Bx/Qr} \sum_{s=1}^{r} \min\left(\frac{x}{hr+s}, \frac{1}{\|sb/r + hr\beta + s\beta\|}\right).$$

Consider first the range $h = 0$ and $1 \leqslant s \leqslant r/2$. Then $|s\beta| \leqslant 1/2R \leqslant 1/2r$, and

$$\left\|\frac{sb}{r} + s\beta\right\| \geqslant \left\|\frac{sb}{r}\right\| - s\,|\beta| \geqslant \left\|\frac{sb}{r}\right\| - \frac{1}{2r} \geqslant \frac{1}{2}\left\|\frac{sb}{r}\right\|.$$

Hence

$$\sum_{1 \leqslant s \leqslant r/2} \min\left(\frac{x}{s}, \frac{1}{\|sb/r + s\beta\|}\right) \ll \sum_{1 \leqslant s \leqslant r/2} \left\|\frac{sb}{r}\right\|^{-1} \ll r \log r. \tag{68}$$

For the other terms, $hr + s \gg (h+1)\, r$.

Fix h and let I be an interval of length $1/r$. We have

$$\left|\left\{1 \leqslant s \leqslant r\colon \left\|\frac{sb}{r} + hr\beta + s\beta\right\| \in I\right\}\right| \leqslant 3, \tag{69}$$

because if the defining property of the above set is satisfied for s_1 and s_2, then

$$\left\|(s_1 - s_2)\frac{b}{r} + (s_1 - s_2)\,\beta\right\| \leqslant \frac{1}{r}.$$

On the other hand, by the triangle inequality, we have

$$\left\|(s_1 - s_2)\frac{b}{r} + (s_1 - s_2)\,\beta\right\| > \left\|\frac{(s_1 - s_2)\, b}{r}\right\| - \frac{1}{r}.$$

Hence $(s_1 - s_2)\, b \equiv 0, \quad \mp 1 \pmod r$, and (69) follows. Therefore, the contribution of terms not counted in (68) is

$$\begin{aligned}
&\ll \sum_{0 \leqslant h \leqslant Bx/Qr} \left(\frac{x}{(h+1)\, r} + \sum_{s=1}^{r-1} \left\| \frac{s}{r} \right\|^{-1} \right) \\
&\ll \frac{x}{r} \log \frac{Bx}{Qr} + \frac{Bx}{Qr}\, r \log r \\
&\ll \frac{x}{r} \log x.
\end{aligned}$$

This, together with (68), proves the lemma.

LEMMA 11. *For $\alpha \in \mathscr{M}(r, b)$, $\alpha = b/r + \beta$, we have*

$$S(x, \alpha) = \frac{\mu(r)}{\phi(r)}\, V(x, \beta) + T(x, \beta) + O(\log r), \tag{70}$$

where

$$V(x, \beta) = \sum_m a\left(\frac{m}{x}\right) e(m\beta),$$

and

$$T(x, \beta) = \frac{1}{\phi(r)} \sum_{\chi \,(\mathrm{mod}\, r)} \tau(\bar{\chi})\, \chi(b)\, \Psi^*(x, \chi, \beta).$$

Proof. It is easily seen that

$$\begin{aligned}
\sum_{\chi \,(\mathrm{mod}\, q)} \chi(c)\, \tau(\bar{\chi}) &= \sum_{h=1}^{q} e\left(\frac{h}{q}\right) \sum_{\chi \,(\mathrm{mod}\, q)} \chi(c)\, \bar{\chi}(h), \\
&= \begin{cases} \phi(q)\, e(c/q) & \text{if} \quad (c, q) = 1 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

We write $\alpha = b/r + \beta$, and by combining the above with (21) obtain

$$\begin{aligned}
S(x, \alpha) = {} & \frac{1}{\phi(r)} \sum_{\chi \,(\mathrm{mod}\, r)} \tau(\bar{\chi})\, \chi(b) \left\{ \sum_m a\left(\frac{m}{x}\right) \Lambda(m)\, \chi(m)\, e(m\beta) \right\} \\
& + 0\left(\sum_{(m, r) > 1} a\left(\frac{m}{x}\right) \Lambda(m) \right).
\end{aligned}$$

In order to complete the proof of the lemma, we use (60) to write this as

$$S(x, \alpha) = \frac{1}{\phi(r)} \sum_{\chi \,(\mathrm{mod}\, r)} \tau(\bar{\chi})\, \chi(b)\, \Psi(x, \chi, b) + 0(\log r)$$

and note that $\tau(\bar{\chi}_0) = \mu(r)$.

Using (70), we can now write

$$\int_0^1 |S(x, \alpha)|^2\, W(x, \alpha)\, da = s_{11} + 0(s_{12} + s_{13} + s_{22} + s_{23} + s_{33}) \tag{71}$$

where

$$s_{11} = \sum_{r \leqslant R} \frac{\mu^2(r)}{\phi^2(r)} \sum_b{}^* \int_{M(r,\, b)} |V(x, \beta)|^2\, W\left(x, \frac{b}{r} + \beta\right) d\beta,$$

$$s_{12} = \sum_{r \leqslant R} \sum_b{}^* \frac{1}{\phi(r)} \int_{M(r,\, b)} \left| V(x, \beta)\, T(x, \beta)\, W\left(x, \frac{b}{r} + \beta\right)\right| d\beta,$$

$$s_{13} = \sum_{r \leqslant R} \sum_b{}^* \frac{\log r}{\phi(r)} \int_{M(r,\, b)} \left| V(x, \beta)\, W\left(x, \frac{b}{r} + \beta\right)\right| d\beta,$$

$$s_{22} = \sum_{r \leqslant R} \sum_b{}^* \int_{M(r,\, b)} \left| T^2(x, \beta)\, W\left(x, \frac{b}{r} + \beta\right)\right| d\beta,$$

$$s_{23} = \sum_{r \leqslant R} \sum_b{}^* \log r \int_{M(r,\, b)} \left| T(x, \beta)\, W\left(x, \frac{b}{r} + \beta\right)\right| d\beta,$$

and

$$s_{33} = \sum_{r \leqslant R} \sum_b{}^* \log^2 r \int_{M(r,\, b)} \left| W\left(x, \frac{b}{r} + \beta\right)\right| d\beta.$$

We now proceed to estimate the error terms in (71).

LEMMA 12. *We have*

$$s_{12} \ll x^{3/2} \log^5 x \tag{72}$$

Proof. By Lemma 10,

$$s_{12} \ll x \log x \sum_{r \leqslant R} \frac{1}{r\phi(r)} \sum_b{}^* \int_{M(r,\, b)} |V(x, \beta)\, T(x, \beta)|\, d\beta \tag{73}$$

By the Cauchy–Schwarz inequality,

$$\int_{M(r,\,b)} |V(x,\beta)\, T(x,\beta)|\, d\beta$$
$$\leqslant \left(\int_{-1/rR}^{1/rR} |V(x,\beta)|\, d\beta\right)^{1/2} \left(\int_{-1/rR}^{1/rR} |T(x,\beta)|^2\, |V(x,\beta)|\, d\beta\right)^{1/2}$$

We now note that

$$V(x,\beta) \ll \min(x, |\beta|^{-1}),$$

which may be obtained by partial summation using the bounded variation and compact support of $a(t/x)$. Hence

$$\int_{-1/rR}^{1/rR} |V(x,\beta)|\, d\beta \ll \int_0^{1/x} x\, d\beta + \int_{1/x}^{1/rR} \frac{1}{\beta}\, d\beta \ll \log x.$$

Consequently,

$$\int_{M(r,\,b)} |V(x,\beta)\, T(x,\beta)|\, d\beta$$
$$\ll \log^{1/2} x \left(\int_{-1/rR}^{1/rR} \min(x, |\beta|^{-1}) |T(x,\beta)|^2\, d\beta\right)^{1/2}. \tag{74}$$

We next have

$$\int_{-1/rR}^{1/rR} \min(x, |\beta|^{-1})\, |T(x,\beta)|^2\, d\beta \ll \int_{1/x}^{2/rR} \left(\frac{1}{\delta^2} \int_0^{\delta} |T(x,\beta)|^2\, d\beta\right) d\delta. \tag{75}$$

This can be seen by observing that the right-hand side of (75) is, on changing the order of integration,

$$= \int_0^{2/rR} |T(x,\beta)|^2 \left(\int_{\max(\beta,\, 1/x)}^{2/rR} \frac{1}{\delta^2}\, d\delta\right) d\beta.$$

By (74) and (75), we can write

$$\sum_b{}^{*} \int_{M(r,\,b)} |V(x,\beta)\, T(x,\beta)|\, d\beta$$
$$\ll (\log x)^{1/2} \sum_b{}^{*} \left(\int_{1/x}^{2/rR} \int_0^{\delta} \frac{1}{\delta^2}\, |T(x,\beta)|^2\, d\beta\, d\delta\right)^{1/2}.$$

We apply the Cauchy–Schwarz inequality to the sum over b and see that the right-hand side of the above is

$$\ll (\phi(r)\log x)^{1/2}\left(\sum_b{}^* \int_{1/x}^{2/rR}\int_0^\delta \frac{1}{\delta^2}|T(x,\beta)|^2\,d\beta\,d\delta\right)^{1/2}.$$

Furthermore,

$$\sum_b{}^* \int_0^\delta |T(x,\beta)|^2\,d\beta \ll r\delta\, x(\log rx)^4$$

by Lemma 9. Therefore

$$\sum{}^* \int_{1/x}^{2/rR}\int_0^\delta \frac{1}{\delta^2}|T(x,\beta)|^2\,d\beta\,d\delta \ll rx(\log x)^5$$

This gives

$$\sum_b{}^* \int_{M(r,b)} |V(x,\beta)\,T(x,\beta)|\,d\beta \ll x^{1/2}r^{1/2}\phi(r)^{1/2}\log^3 x.$$

On appealing to (73), we obtain the assertion of the lemma.

LEMMA 13. *We have*

$$s_{22} \ll x^{3/2}\log^6 x.$$

Proof. By Lemma 10 and the definition of s_{22}, we have

$$s_{22} \ll x\log x \sum_{r\leqslant R}\frac{1}{r}\sum_b{}^* \int_{-1/rR}^{1/rR} |T(x,\beta)|^2\,d\beta.$$

By Lemma 9, the right-hand side of the above is

$$\ll x\log x \sum_{r\leqslant R}\frac{x}{rR}\log^4 rx \ll x^{3/2}\log^6 x.$$

This establishes the lemma.

LEMMA 14. *We have*

$$s_{23} \ll x(\log x)^5.$$

Proof. We observe by Lemma 10 that

$$s_{23} \ll x \log x \sum_{r \leqslant R} \frac{\log r}{r} \sum_b{}^* \int_{M(r,b)} |T(x,\beta)|\, d\beta,$$

by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{M(r,b)} |T(x,\beta)|\, d\beta &\leqslant \left(\int_{-1/rR}^{1/rR} d\beta\right)^{1/2} \left(\int_{-1/rR}^{1/rR} |T(x,\beta)|^2\, d\beta\right)^{1/2} \\ &\ll (rR)^{-1/2} \left(\int_{-1/rR}^{1/rR} |T(x,\beta)|^2\, d\beta\right)^{1/2},\end{aligned}$$

so

$$\sum_b{}^* \int_{M(r,b)} |T(x,\beta)|\, d\beta \ll (rR)^{-1/2} \sum_b{}^* \left(\int_{-1/rR}^{1/rR} |T(x,\beta)|^2\, d\beta\right)^{1/2}.$$

We now apply the Cauchy–Schwarz inequality again, this time to the sum over b. Then the right-hand side of the above is

$$\ll (rR)^{-1/2}\, \phi(r)^{1/2} \left(\sum_b{}^* \int_{-1/rR}^{1/rR} |T(x,\beta)|^2\, d\beta\right)^{1/2}$$

Using Lemma 9, we deduce that this is

$$\ll x^{1/2} \phi(r)^{1/2}\, R^{-1} r^{-1/2}\, (\log rx)^2.$$

Hence

$$s_{23} \ll x^{3/2} \log x \cdot R^{-1} \sum_{r \leqslant R} \frac{\phi(r)^{1/2}}{r^{3/2}} \log r (\log rx)^2$$

$\ll x(\log x)^5$, completing the proof of the lemma.

We now proceed to estimate s_{13} and s_{33}.

LEMMA 15. *We have*

$$s_{13} \ll x \log^4 x,$$

and

$$s_{33} \ll x^{1/2} \log^4 x.$$

Proof. By Lemma 10,

$$s_{13} \ll x \log x \sum_{r \leqslant R} \sum_{b}^{*} \frac{\log r}{r\phi(r)} \int_{M(r,b)} |V(x,\beta)|\, d\beta$$
$$\ll x \log x \sum_{r \leqslant R} \frac{\log r}{r} \cdot \left(\int_{0}^{1/x} x\, d\beta + \int_{1/x}^{2/rR} \frac{1}{\beta}\, d\beta \right)$$
$$\ll x \log^4 x.$$

On the other hand, again by Lemma 10,

$$s_{33} \ll x \log x \sum_{r \leqslant R} \sum_{b}^{*} \frac{\log^2 r}{r} \cdot \frac{1}{rR} \ll x^{1/2} \log^4 x.$$

This completes the proof of the lemma.

We now state our

THEOREM 1. *We have, for* $R \ll Q$,

$$\int_0^1 |S(x,\alpha)|^2\, W(x,\alpha)\, d\alpha$$
$$= \sum_{Q < q \leqslant x} \sum_{1 \leqslant k \leqslant Bx/q} \sum_{\substack{r \leqslant R \\ r \mid k}} \frac{\mu^2(r)}{\phi(r)} A(kq) + O(x^{3/2} \log^6 x) \tag{76}$$

where

$$A(u) = \sum_{n} a\left(\frac{m}{x}\right) a\left(\frac{m+u}{x}\right).$$

Proof. In view of the assertions of Lemmas 12, 13, 14, 15 and (71), we have

$$\int_0^1 |S(x,\alpha)|^2\, W(x,\alpha)\, d\alpha = s_{11} + 0(x^{3/2} \log^6 x). \tag{77}$$

We also have

$$W\left(x, \frac{b}{r} + \beta\right) = \sum_{Q < q \leqslant x} \sum_{\substack{1 \leqslant k \leqslant Bx/q \\ r \mid k}} e\left(-qk\left(\frac{b}{r} + \beta\right)\right)$$
$$+ \sum_{Q < q \leqslant x} \sum_{\substack{1 \leqslant k \leqslant Bx/q \\ r \nmid k}} e\left(-qk\left(\frac{b}{r} + \beta\right)\right)$$
$$= \sum_{Q < q \leqslant x} \sum_{\substack{1 \leqslant k \leqslant Bx/q \\ r \mid k}} e(-qk\beta) + 0\left(\sum_{\substack{1 \leqslant k \leqslant Bx/Q \\ r \nmid k}} \left\| k\left(\frac{b}{r} + \beta\right) \right\|^{-1} \right). \tag{78}$$

Therefore

$$S_{11}=\sum_{Q<q\leqslant x}\ \sum_{1\leqslant k\leqslant Bx/q}\ \sum_{\substack{r\leqslant R\\ r\mid k}}\frac{\mu^2(r)}{\phi^2(r)}\ {\sum_b}^*\int_{M(r,b)}|V(x,\beta)|^2\, e(-qk\beta)\,d\beta$$

$$+O\left(\sum_{1\leqslant k\leqslant Bx/Q}\ \sum_{\substack{r\leqslant R\\ r\nmid k}}\frac{\mu^2(r)}{\phi^2(r)}\ {\sum_b}^*\int_{M(r,b)}|V(x,\beta)|^2\left\|k\left(\frac{b}{r}+\beta\right)\right\|^{-1}d\beta\right). \tag{79}$$

Now for $\beta\in M(r,b)$,

$$\left\|k\left(\frac{b}{r}+\beta\right)\right\|\gg\left\|\frac{bk}{r}\right\|-k\beta\gg\left\|\frac{bk}{r}\right\|-\frac{k}{rR}.$$

Since $k\leqslant Bx/Q$, and $Q\gg R=[x^{1/2}]$, we have $k/rR\leqslant Bx/QrR\leqslant 1/2r$, and hence $\|k(bk/r+\beta)\|\gg\|bk/r\|$, which together with

$$\int_0^1|V(x,\beta)|^2\,d\beta\ll x, \tag{80}$$

shows the error in (79) contributes

$$\ll\sum_{1\leqslant k\leqslant Bx/Q}\ \sum_{\substack{r\leqslant R\\ r\nmid k}}\frac{\mu^2(r)}{\phi^2(r)}\ {\sum_b}^*\left\|\frac{bk}{r}\right\|^{-1}x\ll x\log^3 x.$$

For the main term in (79) we can extend the range of integration to $[0,1]$ with an error

$$\ll\sum_{Q<q\leqslant x}\ \sum_{1\leqslant k\leqslant Bx/q}\ \sum_{\substack{r\leqslant R\\ r\mid x}}\frac{\mu^2(r)}{\phi^2(r)}\ {\sum_b}^*\int_{1/2rR}^{1/2}\frac{1}{\beta^2}\,d\beta$$

$$\ll\sum_{Q<q\leqslant x}\ \sum_{1\leqslant k\leqslant Bx/q}\ \sum_{\substack{r\leqslant R\\ r\mid k}}\frac{\mu^2(r)\,rR}{\phi(r)}$$

$$\leqslant xR\sum_{q\leqslant x}\frac{1}{q}\sum_{r\leqslant R}\frac{\mu^2(r)}{\phi(r)}$$

$$\ll xR\log^2 x\ll x^{3/2}\log^2 x.$$

We conclude

$$S_{11} = \sum_{Q<q\leqslant x} \sum_{1\leqslant k\leqslant Bx/q} \sum_{\substack{r\leqslant R \\ r\mid k}} \frac{\mu^2(r)}{\phi(r)} \int_0^1 |V(x,\beta)|^2 e(-qk\beta)\,d\beta$$

$$+O(x^{3/2}\log^2 x), \tag{81}$$

and the lemma follows on noting that

$$\int_0^1 |V(x,\beta)|^2 e(-qk\beta)\,d\beta = A(kq). \tag{82}$$

4. Evaluation of $J(x; 0, Q)$

For the definitions of $J(x; y, z)$ and $J_i(x; y, z)$ we refer to (47) and (49).

Lemma 16. *We have*

$$J_1(x; y, x) = 2 \sum_{y<q\leqslant x} \sum_{1\leqslant k\leqslant Bx/q} \frac{kA(kq)}{\phi(k)} + O(x^{3/2}\log^6 x). \tag{83}$$

Proof. We may assume $y \gg x^{1/2}\log x$ since the contribution of the terms with $y\leqslant x^{1/2}\log x$ is bounded by $J(x; 0, y)$, which by (47) and Lemma 1 is $\ll yx\log^2 Q$. The factor $\log^2 Q$ arises from the estimate $\sum_\gamma |K(1/2+i\gamma)| \ll \log q$ which follows from the Riemann-von Mangoldt formula for the zeros of $L(s,\chi)$ and the condition that $K(\sigma+it) \times \log(|t|+2) \in L^1$. We note that, under this assumption, since $q>y$, we have $k\leqslant Bx/q < Bx/y < R$, and

$$\sum_{\substack{r\leqslant R \\ r\mid k}} \frac{\mu^2(r)}{\phi(r)} = \sum_{r\mid k} \frac{\mu^2(r)}{\phi(r)} = \prod_{p\mid k}\left(1+\frac{1}{p-1}\right) = \frac{k}{\phi(k)}. \tag{84}$$

The assertion of the lemma can now be obtained using (23) and Theorem 1.

We now proceed to work with the double sum occurring in (83). To this end we prove a lemma concerning the coefficients $a(n/x)$ and their convolution $A(u)$.

LEMMA 17. *For $u \geqslant 0$, we have*

$$a\left(\frac{n+u}{x}\right)=a\left(\frac{n}{x}\right)+O\left(\frac{u}{x}\right), \tag{85}$$

$$\sum_n a\left(\frac{n}{x}\right)=K(1)\,x+O(1), \tag{86}$$

$$A(0)=\kappa x+O(1), \tag{87}$$

$$A(u)=\kappa x+O(u), \tag{88}$$

and

$$\int_0^\infty A(u)\,du=\frac{1}{2}K^2(1)\,x^2+O(x). \tag{89}$$

Proof. Since $a(n+u/x)-a(n/x)=\int_{n/x}^{n/x+u/x} da(u)$, equation (85) follows by the bounded variation of $a(u)$. To prove (86), we write

$$\sum_n a\left(\frac{n}{x}\right)=\int_0^\infty a\left(\frac{t}{x}\right) d([t])=x\int_0^\infty a(u)\,du+\int_0^\infty a(u)\,d([ux]-ux)$$

We then observe that

$$\int_0^\infty a(u)\,du=K(1), \tag{90}$$

by (9) and

$$\int_0^\infty a(u)\,d([ux]-ux) \ll \int_0^\infty ([ux]-ux)\,da(u) \ll 1, \tag{91}$$

by the bounded variation and compact support of $a(u)$. Next, we have

$$A(0)=\sum_n a^2\left(\frac{n}{x}\right)=\int_0^\infty a^2\left(\frac{u}{x}\right) du+O\left(\int_0^\infty a^2\left(\frac{u}{x}\right) d([u]-u)\right). \tag{92}$$

By Plancherel's theorem for the Mellin transform, the first term on the right is κx and the error term is $0(1)$ just as in (91). The assertion (88) now follows on noting that

$$\begin{aligned} A(u)&=\sum_n a\left(\frac{n}{x}\right) a\left(\frac{n+u}{x}\right)\\ &=\sum_n a^2\left(\frac{n}{x}\right)+O\left(\frac{u}{x}\sum_n a\left(\frac{n}{x}\right)\right)\\ &=A(0)+O(u). \end{aligned}$$

To prove (89), we write

$$\begin{aligned}\int_0^\infty A(u)\,du &= \sum_n a\left(\frac{n}{x}\right)\int_0^\infty a\left(\frac{n+u}{x}\right)du\\ &= \sum_n a\left(\frac{n}{x}\right)\left(\sum_{u=0}^{\infty} a\left(\frac{n+u}{x}\right)+O(1)\right)\\ &= \frac{1}{2}\left(\sum_n a\left(\frac{n}{x}\right)\right)^2+O(x)\\ &= \frac{1}{2}K^2(1)\,x^2+O(x)\end{aligned}$$

by (86).

Lemma 18. *Let*

$$D(r)=\sum_{\substack{1\leqslant k\leqslant Bx/q\\ Q<q\leqslant x\\ kq=r}}\frac{k}{\phi(k)}$$

and

$$\varDelta(t)=\sum_{r\leqslant t}D(r)$$

Then for $Q\leqslant t\leqslant Bx$, *we have*

$$\varDelta(t)=\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\,t\log\frac{t}{Q}+c_2t+\frac{1}{2}\,Q\log\frac{t}{Q}+c_3Q+E(t), \tag{93}$$

where c_2 *and* c_3 *are defined as in Lemma* 5 *and* $E(t)\ll Q^{5/4}t^{-1/4}+x/Q$.

Proof. We write

$$\begin{aligned}\varDelta(t) &= \sum_{k\leqslant t/Q}\frac{k}{\phi(k)}\sum_{Q<q\leqslant t/k}1=\sum_{k\leqslant t/Q}\frac{k}{\phi(k)}\left(\left[\frac{t}{k}\right]-Q\right)\\ &= \sum_{k\leqslant t/Q}\left(\frac{t}{\phi(k)}-\frac{Qk}{\phi(k)}\right)+O\left(\frac{t}{Q}\right)\\ &= Q\sum_{k\leqslant t/Q}\frac{1}{\phi(k)}\left(\frac{t}{Q}-k\right)+O\left(\frac{t}{Q}\right).\end{aligned}$$

An appeal to Lemma 5 gives the desired result. We now state our main result concerning $J_1(x;Q,x)$.

LEMMA 19. *For $x^{1/2}\log^6 x \leqslant Q \leqslant x$, we have*

$$J_1(x; Q, x) = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} K^2(1)\, x^2 \log \frac{x}{Q} + c_1 K^2(1)\, x^2 + \kappa Q x \log \frac{x}{Q} + 2\int_Q^\infty A(t) \log \frac{t}{x}\, dt + O(Qx), \tag{94}$$

where c_1 is defined as in Lemma 5.

Proof. By (83), we have

$$J_1(x; Q, x) = 2 \sum_{r > Q} A(r)\, D(r) + O(x^{3/2} \log^6 x) \tag{95}$$

On the other hand,

$$\sum_{r > Q} A(r) D(r) = \int_Q^\infty A(t)\, d(\Delta(t)) \tag{96}$$

The integral above, by Lemma 18, is

$$= \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} \int_Q^\infty A(t) \log t\, dt + \left(\frac{\zeta(2)\,\zeta(3)}{\zeta(6)} + c_2 - \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} \log Q \right) \int_Q^\infty A(t)\, dt + \frac{1}{2} Q \int_Q^\infty \frac{A(t)}{t}\, dt + \int_0^\infty A(t)\, d(E(t)).$$

We now observe that by (88),

$$\int_y^z A(t)\, dt \ll zx \qquad \text{for} \quad Bx \geqslant z > y > 0. \tag{97}$$

We also make use of the following properties of $A(t)$:

$$\int_0^\infty d(A(t)) \ll 1 \tag{98}$$

$$\int_Q^\infty \frac{A(t)}{t}\, dt = \int_Q^x \frac{A(t)}{t}\, dt + \int_x^\infty \frac{A(t)}{t}\, dt = \kappa x \log \frac{x}{Q} + O(x) \tag{99}$$

On using this we arrive at

$$\sum_{r>Q} A(r)\,D(r)=\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\int_Q^\infty A(t)\log\frac{t}{Q}\,dt$$
$$+\left(\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}+c_2\right)\int_0^\infty A(t)\,dt+\frac{1}{2}\kappa Qx\log\frac{x}{Q}+O(Qx) \tag{100}$$

We note that $c_2+(\zeta(2)\,\zeta(3)/\zeta(6))=c_1$. We treat the first integral on the right hand side of (100) as follows:

$$\int_Q^\infty A(t)\log\frac{t}{Q}\,dt=\int_Q^\infty A(t)\log\frac{x}{Q}\,dt+\int_Q^\infty A(t)\log\frac{t}{x}\,dt$$
$$=\frac{1}{2}K^2(1)\,x^2\log\frac{x}{Q}+\int_Q^\infty A(t)\log\frac{t}{x}\,dt+O(Qx)$$

This gives the required result.

Lemma 20. *We have*

$$J_2(x;y,z)=(\kappa x\log x+c_4x)(z-y)+O(zx^{1/2}\log^2 x)+O(x\log^2 z). \tag{101}$$

where

$$c_4=\int_0^\infty a^2(u)\log u\,du \tag{102}$$

Proof.

$$\sum_{(n,q)=1} a^2\left(\frac{n}{x}\right)\Lambda^2(n)=\sum_n a^2\left(\frac{n}{x}\right)\Lambda^2(n)+O(\log^2 q), \tag{103}$$

Since by (9) $a(u)\ll u^{-c}\ll 1$ and

$$\sum_{\substack{n\leqslant Bx\\(n,q)>1}}\Lambda^2(n)\ll\sum_{n\mid q}\Lambda^2(n)\ll\log^2 q.$$

On the other hand,

$$\sum_n a^2\left(\frac{n}{x}\right)\Lambda^2(n)=\int_{Ax}^{Bx} a^2\left(\frac{t}{x}\right)d\left(\sum_{n\leqslant t}\Lambda^2(n)\right).$$

By the prime number theorem and the Riemann hypothesis, this is

$$\int_{Ax}^{Bx} a^2\left(\frac{t}{x}\right) \log t \, dt + O(x^{1/2} \log^2 x).$$

By setting $t = ux$ in the integral above, we obtain

$$\begin{aligned}\sum_{(n,q)=1} a^2\left(\frac{n}{x}\right) \Lambda^2(n) &= x \log x \int_A^B a^2(u)\, du \\ &\quad + x \int_A^B a^2(u) \log u \, du + O(x^{1/2} \log^2 x) + O(\log^2 q) \\ &= \kappa x \log x + c_4 x + O(x^{1/2} \log^2 x) + O(\log^2 q). \quad (104)\end{aligned}$$

We sum (103) over q in the range $y < q \leqslant z$ to obtain the required result.

LEMMA 21. *We have*

$$\begin{aligned} J_3(x; y, z) &= \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} K^2(1)\, x^2 \log \frac{z}{y} + c_1 K^2(1)\, x^2 \\ &\quad + O\left(\frac{x^2 \log y}{y}\right) + O(x^{3/2} \log^2 x \log z) + O(x \log^3 z). \quad (105)\end{aligned}$$

Proof. As in the proof of the previous lemma,

$$\sum_{(n,q)=1} a\left(\frac{n}{x}\right) \Lambda(n) = \sum_n a\left(\frac{n}{x}\right) \Lambda(n) + O(\log q). \quad (106)$$

We also have

$$\sum_n a\left(\frac{n}{x}\right) \Lambda(n) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} K(s) \left\{\frac{-\zeta'(s)}{\zeta(s)}\right\} x^s \, ds,$$

for some $b \in D$, $b > 1$. We move the contour of integration to $\operatorname{Re}(s) = c \in D$, $c < 0$, and deduce that the above is $= K(1)\, x + O(x^{1/2} \log^2 x)$. Hence

$$\begin{aligned}\left(\sum_{(n,q)=1} a\left(\frac{n}{x}\right) \Lambda(n)\right)^2 &= K^2(1)\, x^2 + O(x^{3/2} \log^2 x) \\ &\quad + O(x \log q) + O(\log^2 q). \quad (107)\end{aligned}$$

We appeal to Lemma 5 to obtain (105).

5. Evaluation of $J(x; 0, Q)$.

We now apply our findings to the evaluation of $J(x; 0, Q)$. Our main result in this direction is contained in the following:

LEMMA 22. *We have*

$$J(x;0,Q)=\begin{cases}\kappa Qx\log x+O(Qx)+O(x^2\log Q) & \text{if } \ x\leqslant Q\\ \kappa Qx\log Q+O(Qx\log(2x/Q)) & \text{if } \ x^{1/2}\log^7 x\leqslant Q\leqslant x\end{cases} \tag{108}$$

Proof. We first treat the case $x^{1/2}\log^7 x\leqslant Q\leqslant x$. On appealing to Lemmas 19, 20, and 21 and making use of (48) for $y=Q$ and $z=x$, we conclude

$$J(x;Q,x)=-\kappa Qx\log Q+\kappa x^2\log x+c_4x^2+2\int_Q^\infty A(t)\log\frac{t}{x}\,dt+O(Qx) \tag{109}$$

For $Q_1=x^{1/2}\log^6 x$ we have

$$J(x;0,Q)=J(x;Q_1,x)-J(x;Q,x)+O(Q_1x\log^2 x) \tag{110}$$

where the error term comes from the estimate $J(x;0,y)\ll yx\log^2 Q$ mentioned before the proof of Lemma 16. Hence

$$J(x;0,Q)=\kappa Qx\log Q+2\int_{Q_1}^{Q} A(t)\log\frac{t}{x}\,dt+O(Qx) \tag{111}$$

Appealing to the estimate

$$\int_{Q_1}^{Q} A(t)\log\frac{t}{x}\,dt\ll Qx\log\frac{x}{Q},\ \text{(obtained from (88))} \tag{112}$$

completes the proof of the lemma for this case.

For the case $x\leqslant Q$, we have $J_1(x;Bx,Q)=0$. By Lemma 19, we also have $J_1(x;x^{1/2}\log^6 x,Bx)\leqslant x^2\log x$. Slight modifications of Lemmas 20 and 21 give, respectively

$$J_2(x;0,Q)=\kappa Qx\log x+c_4Qx+O(Qx^{1/2}\log^2 x)+O(Q\log^2 x), \tag{113}$$

$$J_3(x;0,Bx)\ll x^2\log x. \tag{114}$$

Also, we have, trivially,

$$J_3(x; Bx; Q) \ll x^2 \log \frac{Q}{x}. \tag{115}$$

Combining (113), (114), (115) and the estimates on $J_1(x; 0, Q)$ gives the result.

Proof of the Main Theorem. We now proceed to prove our Main Theorem. We have

$$F_K(\alpha, Q) = (\kappa Q \log Q)^{-1} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{\gamma} W_K(\gamma)\, Q^{i\gamma\alpha} \right|^2 \tag{116}$$

where $w_K(\gamma) = K(\frac{1}{2} + i\gamma)$. Writing (24) as $L(x, \chi) = R(x, \chi)$ again we get

$$F(\alpha, Q) = (\kappa Q \log Q)^{-1} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \,(\mathrm{mod}\, q)} |L(Q^\alpha, \chi)|^2 + O\left(\frac{\log^2 Q}{Q}\right), \tag{117}$$

the error above resulting from the omission of zeros belonging to $L(s, \chi_0)$, for the principal character $\chi_0 \,(\mathrm{mod}\, q)$. Indeed, we have, for these zeros

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\gamma, \gamma'} Q^{i\alpha(\gamma - \gamma')} w_K(\gamma)\, w_K(\gamma') \ll \log^3 Q. \tag{118}$$

We next apply the following consequence of the Cauchy–Schwarz inequality to $\sum_{q \leqslant Q} 1/\phi(q) \sum_{\chi \neq \chi_0} |R(Q^\alpha, \chi)|^2$. If $M_k = \sum_{q \leqslant Q} 1/\phi(q) \times \sum_{\chi \neq \chi_0} |A_k(x, \chi)|^2$, and $M_1 \geqslant M_2 \geqslant M_3$, then

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \left| \sum_{k=1}^{3} A_k \right|^2 = M_1 + O((M_1 M_2)^{1/2}).$$

We have three cases. We assume, without loss of generality that $\alpha \geqslant 0$. Let $\eta = 8/3 \log^3 Q(-1 + (1 + \frac{3}{2}\log^2 Q)^{1/2})$.

Case 1. $0 \leqslant \alpha < 2/\log Q - \eta$. Then our M_1 term is given by

$$Q^{-\alpha} \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} a^2(Q^\alpha) \log^2 \frac{q}{\pi}. \tag{119}$$

By a straightforward summation by parts, and using (16) we see that this is

$$\kappa Q(\log Q)\, \delta(\alpha) \cdot \left(1 + 0\left(\frac{1}{\log Q}\right)\right). \tag{120}$$

Case 2. $2/\log Q-\eta \leqslant \alpha \leqslant 1$. Then our M_1 term is $Q^{-\alpha}J(Q^{\alpha}; 0, Q)$ and hence by (108) is

$$\kappa\alpha Q \log Q + O(Q^{\alpha} \log Q). \tag{121}$$

Case 3. $1 \leqslant \alpha \leqslant 2-(14 \log\log Q/\log Q)$. In this case, M_1 term is again $Q^{-\alpha}J(Q^{\alpha}, 0, Q)$ but now equals by (108),

$$\kappa Q \log Q + O(Q). \tag{122}$$

On combining all three cases, we get the result.

6. Proportion of Simple Zeros

Before proving the corollary to our main theorem, we need a lemma.

Lemma 23. *If $1 \leqslant \alpha < 2$ is fixed then*

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma, \gamma'} \left(\frac{\sin(\alpha/2(\gamma-\gamma') \log Q)}{\alpha/2(\gamma-\gamma') \log Q}\right)^2 w_K(\gamma)\, w_K(\gamma')$$
$$\sim \left(1+\frac{1}{3\alpha^2}\right) N_K(Q) \qquad \text{as} \quad Q \to +\infty. \tag{123}$$

Here $w_K(\gamma) = K(\tfrac{1}{2}+i\gamma)$ as before.

Proof. We follow the argument given in [14]. Let

$$r(u) = \left(\frac{\sin \pi\alpha u}{\pi\alpha u}\right)^2 \tag{124}$$

and use the identity

$$\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma, \gamma'} r\left(\frac{(\gamma-\gamma') \log Q}{2\pi}\right) w_K(\gamma)\, w_K(\gamma)$$
$$= N_K(Q) \int_{-\infty}^{+\infty} F(\beta)\, \hat{r}(\beta)\, d\beta \tag{125}$$

where $\hat{r}(\beta)$ is the Fourier transform of r defined as

$$\hat{r}(\beta) = \int_{-\infty}^{+\infty} r(t)\, e(-\beta t)\, dt. \tag{126}$$

In this case

$$\hat{r}(\beta) = \begin{cases} 1/\alpha^2(\alpha - |\beta|) & \text{if} \quad |\beta| < \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{127}$$

We plug in $F(\beta)$ from our main theorem to the right-hand side of (125), and obtain the result by straightforward integration.

Proof of the Corollary. Let m_ρ be the multiplicity of the zero $\rho = \frac{1}{2} + i\gamma$. We count zeros according to multiplicity. In particular

$$\sum_{\gamma} m_\rho w_K^2(\gamma) = \sum_{\substack{\gamma, \gamma' \\ \gamma = \gamma'}} w_K(\gamma)\, w_K(\gamma') \tag{128}$$

for on both sides a given zero is counted with weight $m_\rho^2 w_K^2(\gamma)$. We have

$$\begin{aligned} \sum_{\substack{\gamma \\ \text{simple}}} w_K^2(\gamma) &\geqslant \sum_{\gamma} (2 - m_\rho)\, w_K^2(\gamma) \\ &\geqslant 2 \sum_{\gamma} w_K^2(\gamma) - \sum_{\gamma, \gamma'} \left(\frac{\sin \alpha/2(\gamma - \gamma) \log Q}{\alpha/2(\gamma - \gamma') \log Q} \right)^2 w_K(\gamma)\, w_K(\gamma'). \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma \\ \text{simple}}} w_K^2(\gamma) \\ &\quad \geqslant 2 \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma} w_K^2(\gamma) \\ &\qquad - \sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma, \gamma'} \left(\frac{\sin (\alpha/2(\gamma - \gamma') \log Q)}{\alpha/2(\gamma - \gamma') \log Q} \right)^2 w_K(\gamma)\, w_K(\gamma'). \end{aligned} \tag{129}$$

We take $\alpha = 2 - \delta$ in Lemma 23 and observe that

$$\begin{aligned} &\sum_{q \leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma, \gamma'} \left(\frac{\sin (\alpha/2(\gamma - \gamma') \log Q)}{\alpha/2(\gamma - \gamma') \log Q} \right)^2 w_K(\gamma)\, w_K(\gamma') \\ &\qquad \leqslant \left(\frac{13}{12} + \varepsilon \right) N_K(Q). \end{aligned} \tag{130}$$

To conclude the proof, we appeal to (10) and (129).

Acknowledgment

The author thanks Professor H. L. Montgomery and the referee for valuable suggestions.

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